Voronoi-Diagrams

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Example appl: "post office problem"

Definitions:

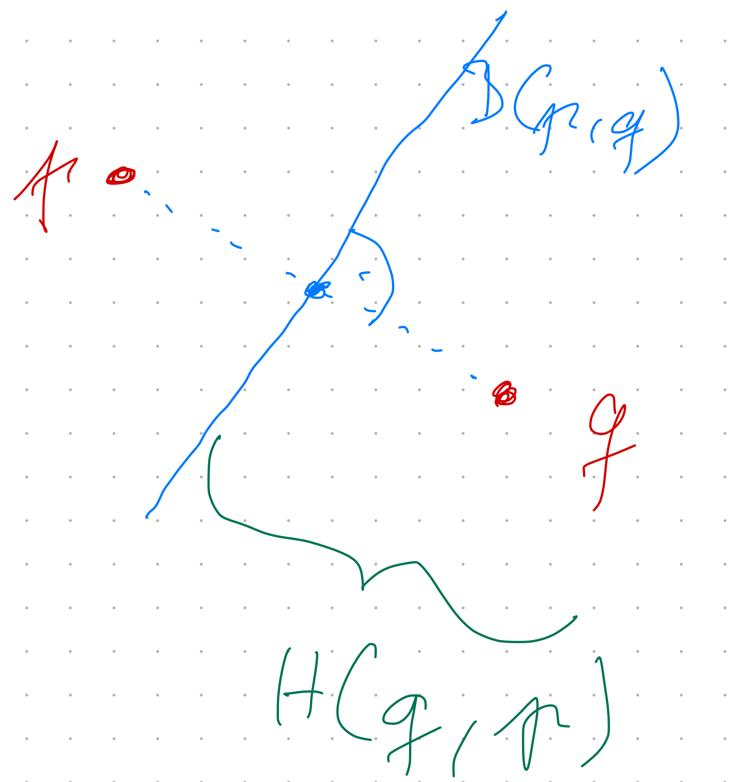
1. Distance metric: $d(p,q) = \|p-q\|$ (other metrics are possible, too)
2. Given $R \subseteq \mathbb{R}^d$, then $\overline{R}$ = closure of R
3. Bisector: given $p, q \in \mathbb{R}^d$

$$B(p,q) := \{ x \mid d(x,p) = d(x,q) \}$$

Note: bisector partitions space into

$$H(p,q) := \{ x \mid d(x,p) < d(x,q) \}$$

$$H(q,p) := \{ x \mid d(x,q) < d(x,p) \}$$



4. Given S = set pts in $\mathbb{R}^d$

Voronoi region of $p \in S$ (wrt. S) is

$$R(p) := \bigcap_{p_i \in S \setminus p} H(p, p_i)$$

5. Voronoi diagram (VD) of S is

$$V(S) := \bigcup_{\substack{p, q \in S \\ p \neq q}} \overline{R(p)} \cap \overline{R(q)}$$

Simple properties:

1) $\forall p \in S$: $R(p)$ is convex

2) $p \neq q \in S \Rightarrow R(p) \cap R(q) = \emptyset$

3) $p, q \in S$, $p \neq q$:

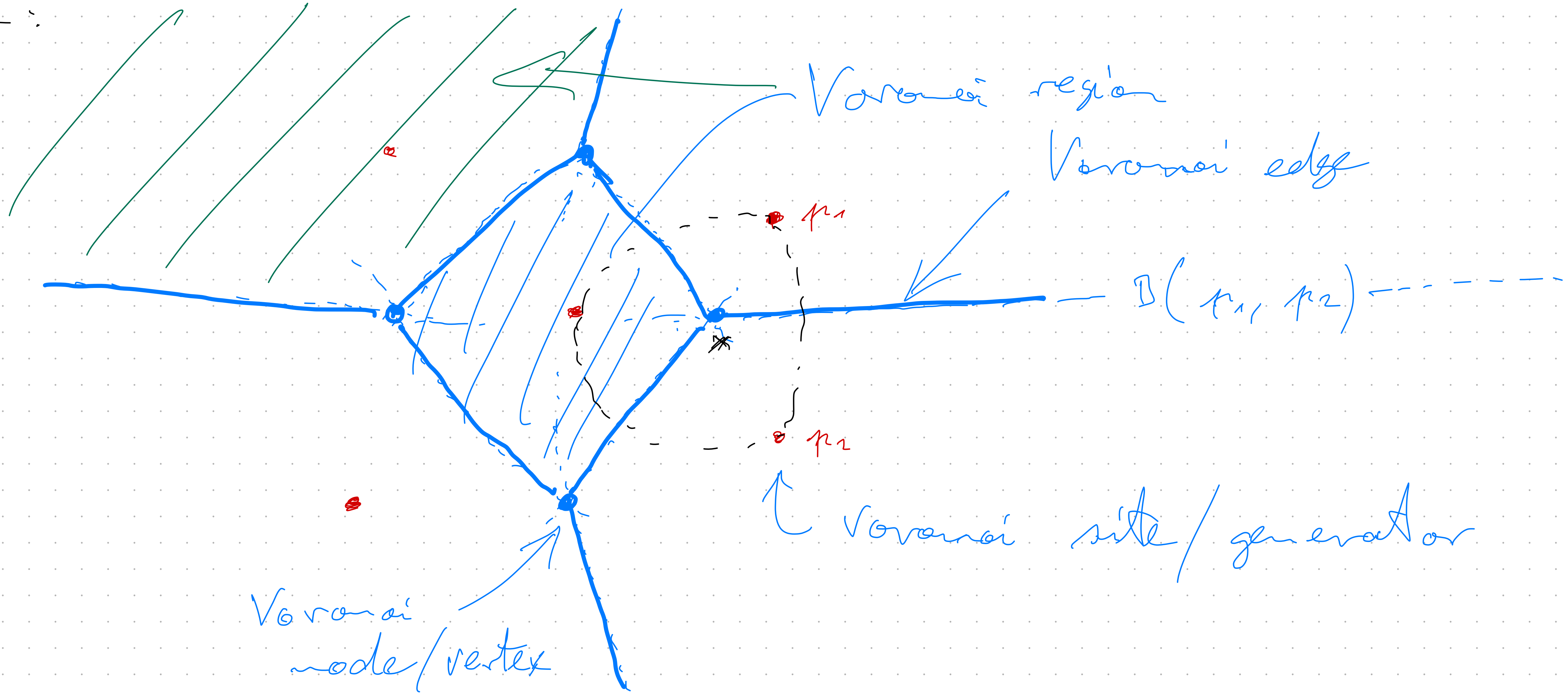
$$\overline{R(p)} \cap \overline{R(q)} \subseteq \overline{H(p,q)} \cap \overline{H(q,p)} = B(p,q)$$

↳ "Voronoi edge" (if at all)

Note: a Voronoi edge does not nesessarily intersect $\overline{pq}$!

4) $R(p)$ = set of all pts closer to p than to any other pt $\in S$

Example:



Lemma: "expanding circle" (2D version)

Let S = set of pts, x any pt;

$C(x)$ = circle around x expanding "slowly".

Three cases:

1. C hits exactly one pt $p \in S$ $\iff$ $x \in R(p)$
2. — " — two pts $p, q \in S$ $\iff$ $x \in$ Voronoi edge on $B(p,q)$
3. — " — 3+ pts $p_1, \dots, p_k$ $\iff$ x is a Voronoi node "between" $R(p_i)$.

Note: "exactly 3 pts" iff S is in general position

Proof:

Case 1) $\Rightarrow$ $d(x,p) = \min_{p_i \in S} d(x, p_i) \iff x \in R(p)$

Case 2) $\Rightarrow$ $x \in B(p,q) \subseteq \overline{H(p,q)}$

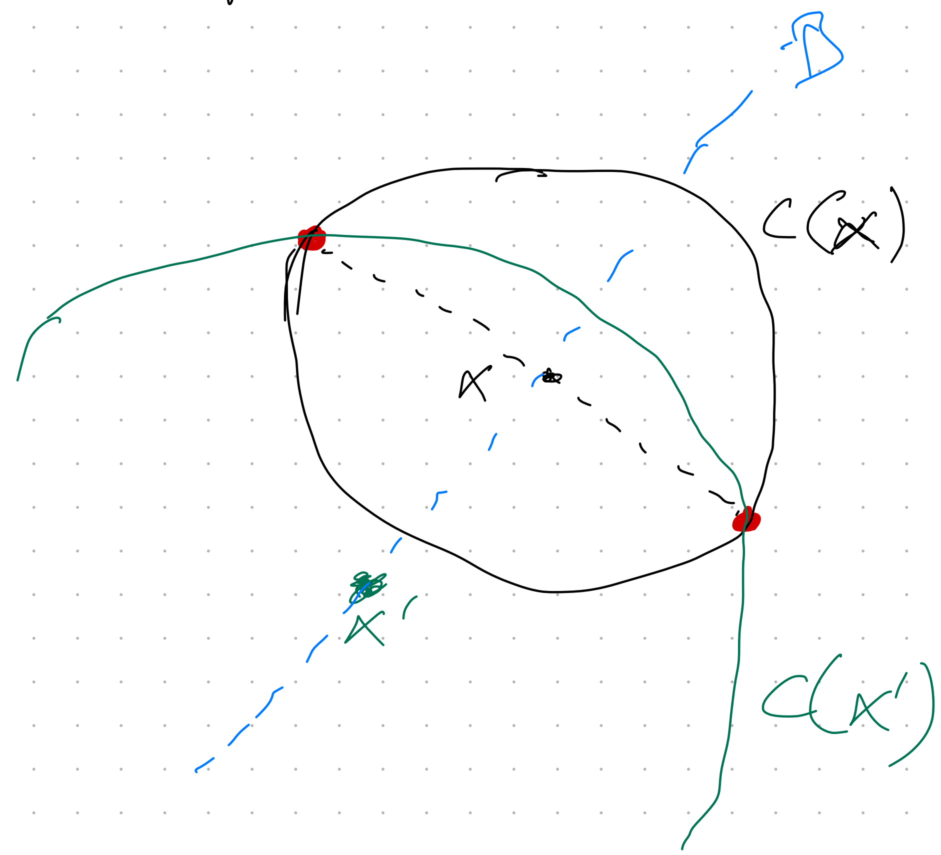
and $\forall r \neq p,q: \; x \in H(p,r)$

$$\Rightarrow x \in \bigcap_{r \neq p} \overline{H(p,r)} = \overline{\bigcap_{r \neq p} H(p,r)} = \overline{R(p)}$$

ok here due to special structure

Analog: $x \in \overline{R(q)}$

$\Rightarrow x \in \overline{R(p)} \cap \overline{R(q)}$



Case 3) $\Rightarrow$ $\forall i = 1, \ldots, k: \; d(x, p_i) = r =$

$$= \min_{q \in S} d(x,q)$$

$\Rightarrow \forall i,j \in \{1..k\}: \; x \in B(p_i, p_j) \Rightarrow$ Voronoi node

Rephrase Lemma of Expanding Circle:

A pt x is on a Voronoi edge $\iff \exists\, C(x)$: $C(x)$ touches exactly 2 pts and there is no other pt from S inside $C(x)$.

A pt x is a Voronoi node $\iff \exists\, C(x)$: ....

Global Properties of $V(S)$

Lemma (connection between $V(S)$ and $CH(S)$):

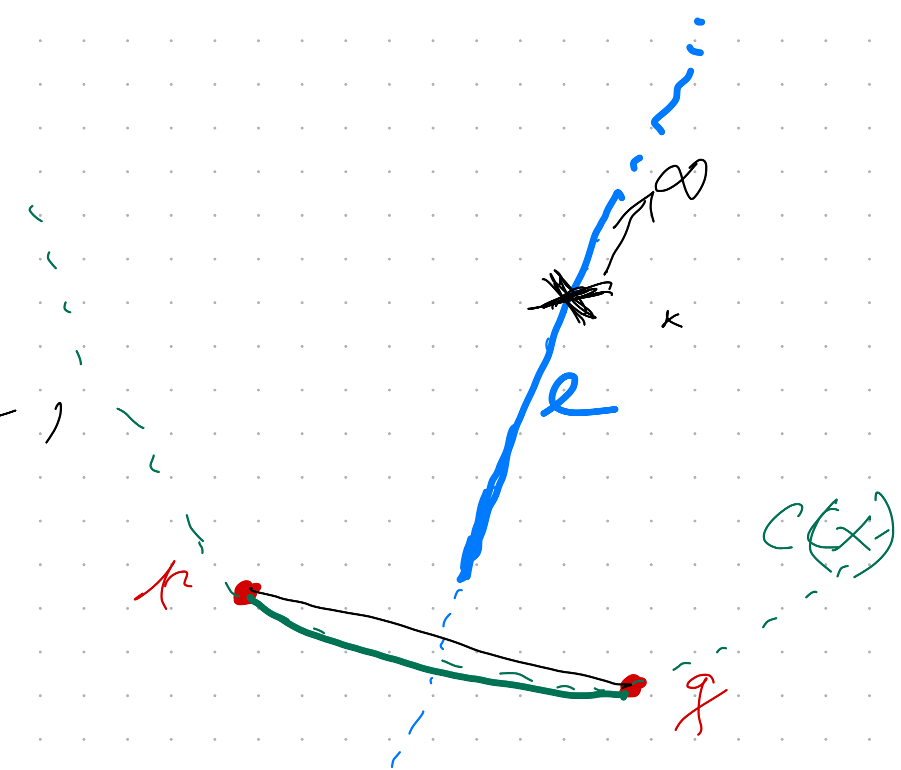
$R(p)$ is unbounded $\iff$ p is on the border of $CH(S)$
(actually a vertex of $CH(S)$).

Proof:

"$\Rightarrow$": $R(p)$ unbounded $\Rightarrow$ $R(p)$ has unbounded edge e,

consider $C(x)$ through p, q,

let $x \to \infty$



⇒ segment of $C(x)$ between p, q approaches $\overline{pq}$.

$C(x)$ never contains a pt $\in S$ (other than p, q)

⇒ definition of edge of CH

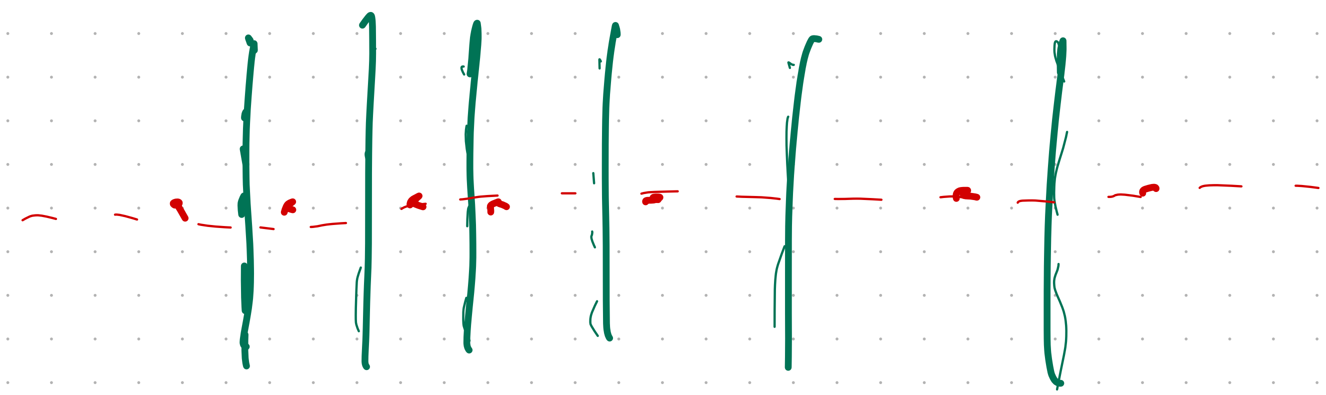
"⇐": let $p, q \in CH(S)$, $\overline{pq}$ edge of CH

⇒ exist $C(x)$ through p, q that does not contain any other pt in S (b/c S is finite and in general pos.)

→ make $C(x)$ bigger → claim

Lemma (w/o proof):

$V(S)$ is always a connected graph, except where all pts in S are on a single line.



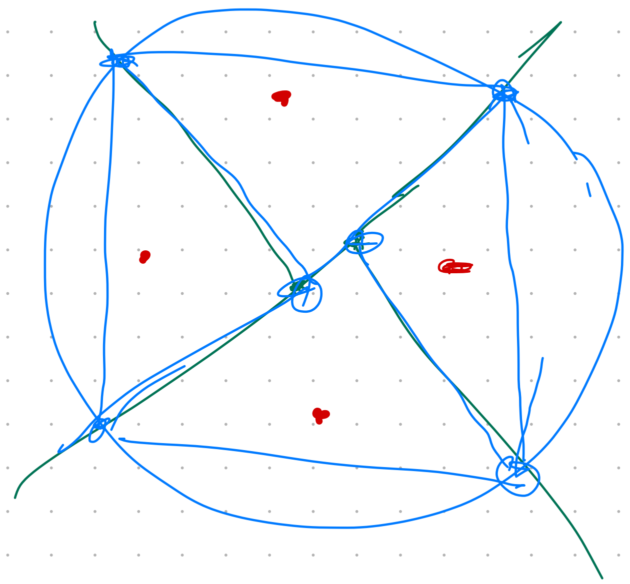
Lemma: (complexity)

The $V(S)$ of n pts in the plane (!) has $O(n)$ many nodes, edges, and regions.

Proof:

Prune unbounded edges by a circle "big enough", replace circle's segment by straight edges

→ apply Euler $(V-E+F=1)$



Theorem:

The construction of $V(S)$ in the plane takes at least $\Omega(n \log n)$.

Proof:

Reduce CH to VD.

Note: $CH(S)$ can be derived from $V(S)$ in $O(n)$.